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On the Degree of Polynomial Approximation in $E^p(D)$

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1. INTRODUCTION

Let D be a Jordan domain in $\mathbb{C}$ with rectifiable boundary Γ. By definition

$$f \in E^p(D) \qquad \text{if and only if} \qquad f \circ \varphi \cdot (\varphi')^{1/p} \in H^p,$$

where φ is a Riemann mapping of the unit disc U onto D. We shall denote by $|\Gamma|$ the length of Γ.

Supplied with the norm

$$\|f\|_p = \left((1/|\Gamma|) \int_\Gamma |f(z)|^p \, |dz|\right)^{1/p},$$

$E^p(D)$ becomes a Banach space for $p \geqslant 1$. For further properties of $E^p(D)$ see, e.g., [3].

The degree of polynomial approximation in $E^p(D)$, $p \geqslant 1$, has been studied by several authors. In [10] Walsh and Russell gave results when Γ is an analytic curve. Later these results were extended to more general domains, for $p > 1$ by Al'per [1] and for $p = 1$ by Andraško [2] and Galan [4]. However, no corners were allowed. In [7] Kokilašvili stated theorems for $p > 1$ that also cover cases when D has corners. Similar results are given in [6].

The results in [7] rely on the boundedness of the operator S: $L^p(\Gamma) \to L^p(\Gamma)$ defined by

$$Sf(z) = \int_\Gamma (f(\zeta)/(\zeta - z)) \, d\zeta, \qquad z \in \Gamma.$$

However, the boundedness is needed only for a certain subspace of $L^p(\Gamma)$. In this paper we shall give a method that enables us to include domains with corners in the case $p = 1$. Since no proofs have been given in [7] we shall also include the case $p > 1$.

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2. Polynomial Expansion in $E^p(D)$

Let $\Psi: S^2\backslash U \to S^2\backslash D$ be the Riemann mapping with $a = \Psi'(\infty) > 0$. In the sequal let q be the conjugate exponent of p, i.e.,

$$p^{-1} + q^{-1} = 1.$$

For $k = 0, 1,\ldots,$ and $R > 1$ let

$$F_{p,k}(z) = \frac{1}{2\pi i} \int_{|w|=R} \frac{w^k[\Psi'(w)]^{1/q}}{\Psi(w) - z}\, dw, \qquad z \in D.$$

Obviously $F_{p,k}$ is a polynomial of degree k. We shall refer to these polynomials as the p-Faber polynomials. To each $f \in E^p(D)$ we associate its p-Faber series

$$f \sim \sum_0^\infty a_k F_{p,k}\,,$$

where

$$a_k = a_k(f) = (2\pi i)^{-1} \int_{|w|=1} f \circ \Psi(w) \cdot [\Psi'(w)]^{1/p}\, w^{-k-1}\, dw.$$

Series of this type were studied for $p > 1$ by Kokilašvili.

Lemma 2.1. *For $n = 0, 1,\ldots,$ the p-Faber coefficients of $F_{p,n}$ are*

$$\begin{aligned} a_k(F_{p,n}) &= 0 \qquad \text{for} \quad k \neq n \\ &= 1 \qquad \text{for} \quad k = n. \end{aligned}$$

Proof. With obvious modifications we can use the proof by Kövari and Pommerenke [8] of the corresponding lemma in the uniform case.

The lemma can be used to prove the following result that will be useful later.

Proposition 2.2. *For every $f \in E^p(D)$, $p \geqslant 1$, the Abel sum of its p-Faber series converges pointwise to f in D.*

Proof. Let $R > 1$. Then for $z \in D$

$$\begin{aligned} &\frac{1}{2\pi i} \int_{|w|=1} \frac{f \circ \Psi(w) \cdot [\Psi'(w)]^{1/p} \cdot [\Psi'(Rw)]^{1/q}}{\Psi(Rw) - z}\, dw \\ &= \sum_0^\infty R^{-k-1} F_{p,k}(z) \cdot (2\pi i)^{-1} \int_{|w|=1} f \circ \Psi(w) \cdot [\Psi'(w)]^{1/p} \cdot w^{-k-1}\, dw \\ &= R^{-1} \sum_0^\infty R^{-k} a_k F_{p,k}(z), \end{aligned}$$

where a_k are the p-Faber coefficients of f. Since $[\Psi'(1/\cdot)]^{1/q} \in H^q$ and Ψ is continuous on $|w| \geqslant 1$ we get

$$\lim_{R\to 1+} \frac{1}{2\pi i} \int_{|w|=1} \frac{f \circ \Psi(w) \cdot [\Psi'(w)]^{1/p} \cdot [\Psi'(Rw)]^{1/q}}{\Psi(Rw) - z}\, dw$$
$$= \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = f(z)$$

for each $z \in D$.

COROLLARY 1. *No two different functions in $E^p(D)$ have the same p-Faber series.*

COROLLARY 2. *If a p-Faber series $\sum a_k F_{p,k}$ is Abel summable in $E^p(D)$, then its sum is the only function in $E^p(D)$ with these p-Faber coefficients.*

Proof. That its sum has $(a_k)_0^\infty$ as p-Faber coefficients follows from Lemma 2.1. This completes the proof.

3. AN OPERATOR FROM H^p INTO $E^p(D)$

Let $\Pi_n(D)$ and Π_n denote the polynomials of degree not exceeding n, considered as subspaces of $E^p(D)$ and H^p, respectively. Further, we let $\Pi(D)$ and Π be the corresponding sets with no restrictions on the degrees.

From the previous section we see that for $p \geqslant 1$ we can define an operator $T_p : \Pi \to E^p(D)$ by

$$(T_p P)(z) = \frac{1}{2\pi i} \int_{|w|=1} \frac{P(w) \cdot [\Psi'(w)]^{1/p}}{\Psi(w) - z}\, dw$$

for $z \in D$. For $p > 1$ we can use generalizations of the M. Riesz inequality to more general domains (see [5]) to prove that T_p is bounded for a wide class of domains. These are the domains considered by Kokilašvili. This shows that the following definition is not empty for $p > 1$.

DEFINITION OF TYPE A_p. Let $p \geqslant 1$. A Jordan domain D with rectifiable boundary is of type A_p if the operator T_p is bounded.

In Section 5 we shall give a sufficient condition for D to be of type A_1. This condition will also permit corners.

If D is of type A_p the linear operator T_p can be extended to the whole of H^p. In fact we have the following

THEOREM 3.1. *Let D be of type A_p. Then there exists a continuous linear operator*

$$T_p : H^p \to E^p(D)$$

such that

(i) $T_p(w^k) = F_{p,k}$ *for* $k = 0, 1, \ldots.$

(ii) T_p *is injective. If $p > 1$ it is moreover surjective.*

Proof. All that remains to prove is (ii). Let $g \in H^p$ with $T_p g = 0$ have the representation

$$g(w) = \sum_0^\infty a_k w^k.$$

Since $g_r = g(r\cdot) \to g$ in H^p as $r \to 1-$ we get

$$\lim_{r \to 1-} T_p g_r = T_p g = 0.$$

But

$$T_p g_r = \sum_0^\infty a_k r^k F_{p,k}$$

and Lemma 2.1 imply

$$a_k = \lim_{r \to 1-} a_k(T_p g_r) = a_k(T_p g) = 0$$

for $k = 0, 1, \ldots,$ where as before, $a_k(T_p g_r)$ is the corresponding p-Faber coefficient of $T_p g_r$. Hence $g = 0$ and T_p is injective.

For $f \in E^p(D)$ and $|u| < 1$ let

$$\tilde{f}(u) = \frac{1}{2\pi i} \int_{|w|=1} \frac{f \circ \Psi(w) \cdot [\Psi'(w)]^{1/p}}{w - u}\, dw.$$

This is the unique holomorphic function in U whose Taylor coefficients at the origin coincide with the p-Faber coefficients of f.

The first part of the proof shows that $g \sim \sum a_k w^k$ implies $T_p g \sim \sum a_k F_{p,k}$. Corollaries 1 and 2 of Proposition 2.2 then give $f \in T_p(H^p)$ if and only if $\tilde{f} \in H^p$ and that $T_p^{-1} f = \tilde{f}$. By the M. Riesz inequality $\tilde{f} \in H^p$ for all $f \in E^p(D)$ if $p > 1$. Hence T_p is surjective in that case.

Remark. From the above it follows in fact that T_p^{-1} is continuous for any Jordan domain with rectifiable boundary in case $p > 1$.

4. The Degree of Approximation

For $f \in E^p(D)$ let

$$E_{p,n}(f) = \inf\{\|f - p\|: p \in \Pi_n(D)\}, \qquad n = 0, 1, \ldots .$$

By Theorem 3.1 all estimates of the degree of approximation in H^p can be transferred to $E^p(D)$ if D is of type A_p. In case $p > 1$ the estimates are in fact equivalent, by the remark after Theorem 3.1.

For $g \in H^p$, $p \geqslant 1$, and $h > 0$ let

$$\omega_p(g, h) = \sup\{\| g(\cdot e^{ix}) - g(\cdot)\|_p : |x| < h\}.$$

The H^p-version of Jackson's theorem states that

$$E_{p,n}(g) \leqslant C\omega_p(g, n^{-1})$$

for $g \in H^p$ and $n = 1, 2,\ldots$. Using Theorem 3.1 and remembering the notation

$$(T_p^{-1}f)(u) = \tilde{f}(u) = \frac{1}{2\pi i}\int_{|w|=1} \frac{f \circ \Psi(w) \cdot [\Psi'(w)]^{1/p}}{w - u}\, dw$$

for $f \in E^p(D)$ and $|u| < 1$, the following result is therefore immediate.

THEOREM 4.2. *Let $p \geqslant 1$ and D be of type A_p. Then for $f \in E^p(D)$ and $n = 1, 2,\ldots$*

$$E_{p,n}(f) \leqslant \| T_p \| E_{p,n}(\tilde{f}) \leqslant C \| T_p \| \omega_p(\tilde{f}, n^{-1}).$$

Remark 1. The remark after Theorem 3.1 yields the inequality

$$E_{p,n}(\tilde{f}) \leqslant C_p E_{p,n}(f)$$

for *any* Jordan domain with rectifiable boundary and $p > 1$. From this, inverse theorems can easily be deduced.

Remark 2. In general $\tilde{f}$ may be much smoother than $f \circ \Psi \cdot (\Psi')^{1/p}$, which appears in the corresponding estimate in [7].

Remark 3. For $p = 1$ the theorem is obviously of no value unless $\tilde{f} \in H^1$.

5. A Sufficient Condition for D to Be of Type A_1

For $p = 1$ difficulties arise because of the failure of the M. Riesz inequality. Our starting point is the same as that of Pommerenke [9].

Let $\rho > 1$. With $|u| = 1$ fixed and $\lambda = (\rho + 1)/2$ we have

$$\log \frac{\Psi(\tau) - \Psi(\lambda u)}{a\tau} = \frac{1}{\pi} \int_{|w|=1} \frac{\rho}{\tau - \rho w} \arg \left[\frac{\Psi(\rho w) - \Psi(\lambda u)}{a\rho w}\right] dw$$

for $|\tau| > \rho$. Remember that $a = \Psi'(\infty)$. We observe that, for a fixed ρ, it is possible to define the branches so that $\arg((\Psi(\rho w) - \Psi(\lambda u))/a\rho w)$ is locally continuously differentiable with respect to u along the circle $|u| = 1$. We may then differentiate with respect to u inside the integral. Hence, we get

$$\frac{\Psi'(\lambda u)}{\Psi(\tau) - \Psi(\lambda u)} = \frac{1}{\pi} \int_{|w|=1} \frac{\rho}{\tau - w} \left(-\frac{1}{\lambda}\frac{\partial}{\partial u} \arg[\Psi(\rho w) - \Psi(\lambda u)]\right) dw. \quad (1)$$

This may serve as a motivation of the following theorem, which is the main result in this paper.

THEOREM 5.1. *A Jordan domain with rectifiable boundary is of type* A_1 *if*

$$\liminf_{\rho\to 1+} \left\| \int_{|u|=1} |(\partial/\partial u)(\arg[\Psi(\rho w) - \Psi(\lambda u)])| \, |du| \right\|_\infty = C_1 < \infty$$

where $\|\cdot\|_\infty = \text{ess sup}_{|w|=1} |\cdot|$.

Proof. Let the domain D fulfill the conditions in the theorem. For $\rho > 1$ we can define linear operators

$$\tilde{T}^\rho: H^1 \to L^1 \text{ (on the unit circle)}$$

by

$$(\tilde{T}^\rho f)(u) = -\frac{1}{\pi\lambda} \int_{|w|=1} f(w) \frac{\partial}{\partial u} (\arg[\Psi(\rho w) - \Psi(\lambda u)]) \, dw.$$

By Fubini's theorem we get

$$\| \tilde{T}^\rho f\|_1 < (1/\pi) \left\| \int_{|u|=1} |(\partial/\partial u)(\arg[\Psi(\rho w) - \Psi(\lambda u)])| \, |du| \right\|_\infty \cdot \|f\|_1 .$$

From the definition of the 1-Faber polynomials it follows that

$$\frac{\Psi'(\lambda u)}{\Psi(\tau) - \Psi(\lambda u)} = \sum_0^\infty \frac{F_{1,k} \circ \Psi(\lambda u) \cdot \Psi'(\lambda u)}{\tau^{k+1}}, \qquad |\tau| > \lambda.$$

Moreover, (1) gives

$$\frac{\Psi'(\lambda u)}{\Psi(\tau) - \Psi(\lambda u)} = \tilde{T}^\rho \left(\frac{\rho}{\tau - w}\right)(u) = \sum_0^\infty (\rho/\tau)^{k+1} \, \tilde{T}^\rho(w^k)(u)$$

for $|\tau| > \rho$. This implies

$$\tilde{T}^{\rho}(w^k)(u) = \rho^{-k-1} \cdot F_{1,k} \circ \Psi(\lambda u) \cdot \Psi'(\lambda u) \tag{2}$$

for $k = 0, 1, \ldots$.

Since the right-hand side of (2) has a limit in L^1 sense as $\rho \to 1+$, the limit

$$\lim_{\rho \to 1+} \tilde{T}^{\rho} P = \tilde{T} P$$

exists for all polynomials P. Furthermore we know that

$$\liminf_{\rho \to 1+} \| \tilde{T}^{\rho} \| = C.$$

Consequently

$$\| \tilde{T} P \|_1 \leqslant C \| P \|_1$$

for all polynomials P. Since moreover

$$\tilde{T}(w^k) = F_{1,k} \circ \Psi \cdot \Psi',$$

we see that for all polynomials P

$$(T_1 P) \circ \Psi \cdot \Psi' = \tilde{T} P.$$

Hence

$$\| T_1 P \|_1 \leqslant C \| P \|_1$$

and thus D is of type A_1.

Remark. Geometrically the condition in the theorem means that the variations of the directions of secants with one fixed endpoint have to be bounded by a constant, independent of these endpoints. If, for instance, the boundary is sufficiently smooth between a finite number of corners, the domain is obviously of type A_1.

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